

NON-AUTONOMOUS SVINOLUPOV JORDAN KdV SYSTEMS

Metin Gürses

Department of Mathematics, Faculty of Sciences
Bilkent University, 06533 Ankara - Turkey

Atalay Karasu and Refik Turhan

Department of Physics, Faculty of Arts and Sciences
Middle East Technical University, 06531 Ankara-Turkey

February 8, 2008

Abstract

Non-autonomous Svinolupov -Jordan KdV systems are considered. The integrability criteria of such systems are associated with the existence of recursion operators. A new non-autonomous KdV system is obtained and its recursion operator is given for all N . The examples for $N = 2$ and $N = 3$ are studied in detail. Some possible transformations are also discussed which map some systems to autonomous cases.

There is recently an increasing interest in the study of integrable nonlinear partial differential equations on associative and non-associative algebras [1] and in their recursion operators [2], [3]. It is well known that one class of integrable autonomous multi-component KdV equations (Korteweg-de Vries) , associated with a Jordan algebra J (commutative and non-associative),

$$q_t^i = q_{xxx}^i + s_{jk}^i q^j q_x^k, \quad s_{jk}^i = s_{kj}^i, \quad i, j, k = 1, 2, \dots, N \quad (1)$$

has been considered by Svinolupov [4] where q^i are real and depend on the variables x and t . The constant parameters s_{jk}^i are structure constants ,with respect to some basis e_i ,of a Jordan algebra J

$$e_i \circ e_j = s_{ij}^k e_k \quad (2)$$

satisfy the commutativity identities

$$s_{pr}^k F_{ljk}^i + s_{jr}^k F_{lpk}^i + s_{jp}^k F_{lrk}^i = 0, \quad (3)$$

where

$$F_{plj}^i = s_{jk}^i s_{lp}^k - s_{lk}^i s_{jp}^k, \quad (4)$$

is the associator of the Jordan algebra. The integrability criteria of the multi-component Jordan KdV system (JKdV)(1) are associated with the existence of higher symmetries and the corresponding recursion operator.

Theorem 1: (Svinolupov) *Let s_{jk}^i be the structure constants of a Jordan algebra, i.e., satisfy the identities (3). The system (1) possesses a recursion operator of the form*

$$\mathcal{R}_j^i = \delta_j^i D^2 + \frac{2}{3} s_{jk}^i q^k + \frac{1}{3} s_{jk}^i q_x^k D^{-1} + \frac{1}{9} (s_{jm}^i s_{kl}^m - s_{km}^i s_{jl}^m) q^l D^{-1} q^k D^{-1}. \quad (5)$$

We only need to prove that \mathcal{R} satisfies the integrability condition [5]

$$\mathcal{R}_{j,t}^i = K_k'^i \mathcal{R}_j^k - \mathcal{R}_k^i K_j'^k, \quad (6)$$

with respect to (3) where $K_k'^i$ is the Fréchet derivative of the system (1). Therefore, the existence of the recursion operator ensures that the system (1) possesses an infinite series of symmetries.

Svinolupov established a one to one correspondence between Jordan algebras and the subsystems(reducible,irreducible,completely reducible) of the system (1).

Definition 1: A system of type (1) is called reducible (triangular) if it decouples into the form

$$U_t^i = F^i(U^k, U_x^k, U_{xxx}^k), \quad i, k = 1, 2, \dots, K, \quad 0 < K < N \quad (7)$$

$$V_t^i = G^i(U^k, U_x^k, V^k, V_x^k, V_{xxx}^k), \quad i = 1, 2, \dots, N - K. \quad (8)$$

under a certain linear transformation which leaves the system (1) invariant. If not, it is irreducible. A system is called completely reducible if the second equation given above does not contain the dynamical variables U^i and U_x^i .

Example 1: For $N = 2$, the complete classification, with respect to Jordan algebra, was given by Svinolupov [6].

$$u_t = u_{xxx} + 2c_0 u u_x, \quad v_t = v_{xxx} + c_0 (uv)_x, \quad (9)$$

$$u_t = u_{xxx} + c_0 u u_x, \quad v_t = v_{xxx} + c_0 (uv)_x, \quad (10)$$

$$u_t = u_{xxx}, \quad v_t = v_{xxx} + c_0 u u_x, \quad (11)$$

where c_0 is an arbitrary constant. The reducible systems (9) and (10) correspond to the JKdV and trivially JKdV (associator is zero) respectively. The last system is completely reducible system.

Example 2: For $N = 3$,

i) The system

$$\begin{aligned} u_t &= u_{xxx} - c_0 (u^2 - v^2 - w^2)_x, \\ v_t &= v_{xxx} - c_0 (uv)_x, \\ w_t &= w_{xxx} - c_0 (uw)_x \end{aligned} \quad (12)$$

is the only irreducible JKdV system [6],[7].

ii) A reducible JKdV system is

$$\begin{aligned}
u_t &= u_{xxx} - 2c_0 u u_x, \\
v_t &= v_{xxx} - c_0 (uv)_x, \\
w_t &= w_{xxx} - c_0 (uw)_x.
\end{aligned} \tag{13}$$

In this work we will investigate the non-autonomous Svinolupov JKdV systems. For this purpose, we consider the non-autonomous form of the system (1) as

$$q_t^i = q_{xxx}^i + s_{jk}^i(t) q^j q_x^k, \quad s_{jk}^i(t) = s_{kj}^i(t), \quad i, j, k = 1, 2, \dots, N \tag{14}$$

where $s_{jk}^i(t)$ are sufficiently differentiable functions. In particular, for $N = 1$ the system (14) is the well known cylindrical KdV (cKdV) equation [8]

$$u_t = u_{xxx} + \frac{6}{\sqrt{t}} u u_x, \tag{15}$$

which possesses a recursion operator [9]

$$\mathcal{R} = tD^2 + 4\sqrt{t}u + \frac{1}{3}x + \frac{1}{6}(12\sqrt{t}u_x + 1)D^{-1}. \tag{16}$$

We are now in a position to propose a recursion operator for the integrability of the system (14). Moreover, motivated by the results obtained in Refs.([4], [6]) and ([9]-[11]) we may state the following theorem.

Theorem 2: *Let s_{jk}^i be the structure constants of a Jordan algebra, i.e., satisfy the identities (3). The system (14) possesses a recursion operator of the form.*

$$\begin{aligned}
\mathcal{R}_j^i &= t\delta_j^i D^2 + \frac{2}{3}\sqrt{t}s_{jk}^i q^k + \frac{1}{3}\delta_j^i x + \left(\frac{1}{3}\sqrt{t}s_{jk}^i q_x^k + \frac{1}{6}\delta_j^i\right) D^{-1} \\
&\quad + \frac{1}{9}F_{lkj}^i q^l D^{-1} q^k D^{-1}.
\end{aligned} \tag{17}$$

We only need to prove that \mathcal{R} satisfies the integrability condition (6) with respect to (3).

Example 3: For $N = 2$,

i) The system

$$\begin{aligned} u_t &= u_{xxx} + \frac{2c_0}{\sqrt{t}}uu_x, \\ v_t &= v_{xxx} + \frac{c_0}{\sqrt{t}}(uv)_x, \end{aligned} \quad (18)$$

is the non-autonomous JKdV where c_0 is an arbitrary constant. The recursion operator \mathcal{R} for the above system is

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_0^0 & \mathcal{R}_1^0 \\ \mathcal{R}_0^1 & \mathcal{R}_1^1 \end{pmatrix}. \quad (19)$$

with

$$\begin{aligned} \mathcal{R}_0^0 &= tD^2 + \frac{1}{3}x + \frac{4c_0}{3}\sqrt{t}u + \frac{1}{6}(4c_0\sqrt{t}u_x + 1)D^{-1}, \\ \mathcal{R}_1^0 &= 0, \\ \mathcal{R}_0^1 &= \frac{2c_0}{3}\sqrt{t}v + \frac{c_0}{3}\sqrt{t}v_xD^{-1} - \frac{c_0^2}{9}uD^{-1}vD^{-1}, \\ \mathcal{R}_1^1 &= tD^2 + \frac{1}{3}x + \frac{2c_0}{3}\sqrt{t}u + \frac{1}{6}(2c_0\sqrt{t}u_x + 1)D^{-1} + \frac{c_0^2}{9}uD^{-1}uD^{-1}. \end{aligned} \quad (20)$$

ii) The non-autonomous reducible JKdV is

$$\begin{aligned} u_t &= u_{xxx} + \frac{c_1}{\sqrt{t}}uu_x, \\ v_t &= v_{xxx} + \frac{c_1}{\sqrt{t}}(uv)_x, \end{aligned} \quad (21)$$

where c_1 is an arbitrary constant. The recursion operator for this system is

$$\begin{aligned} \mathcal{R}_0^0 &= tD^2 + \frac{1}{3}x + \frac{2c_1}{3}\sqrt{t}u + \frac{1}{6}(2c_1\sqrt{t}u_x + 1)D^{-1}, \\ \mathcal{R}_1^0 &= 0, \\ \mathcal{R}_0^1 &= \frac{2c_1}{3}\sqrt{t}v + \frac{c_1}{3}\sqrt{t}v_xD^{-1}, \\ \mathcal{R}_1^1 &= tD^2 + \frac{1}{3}x + \frac{2c_1}{3}\sqrt{t}u + \frac{1}{6}(2c_1\sqrt{t}u_x + 1)D^{-1}. \end{aligned} \quad (22)$$

Example 4: For $N = 3$

i) The non-autonomous irreducible JKdV system is

$$\begin{aligned} u_t &= u_{xxx} - \frac{c_0}{\sqrt{t}}(u^2 - v^2 - w^2)_x, \\ v_t &= v_{xxx} - \frac{c_0}{\sqrt{t}}(uv)_x, \\ w_t &= w_{xxx} - \frac{c_0}{\sqrt{t}}(uw)_x. \end{aligned} \quad (23)$$

ii) The non-autonomous reducible J KdV system

$$\begin{aligned} u_t &= u_{xxx} - \frac{2c_0}{\sqrt{t}}uu_x, \\ v_t &= v_{xxx} - \frac{c_0}{\sqrt{t}}(uv)_x, \\ w_t &= w_{xxx} - \frac{c_0}{\sqrt{t}}(uw)_x, \end{aligned} \quad (24)$$

is the extension of (9). The recursion operators for the systems (23) and (24) are too long, hence we do not give them here.

Finally, we establish linear transformations between autonomous and non-autonomous systems. In the scalar case, the KdV and cKdV equations are equivalent since their solutions are related by simple Lie-point transformation [12]-[16].

$$u(x, t) = t^{-1/2}u'(xt^{-1/2}, -2t^{-1/2}) - \frac{1}{12}xt^{-1/2}. \quad (25)$$

Here we present a generalization of this result to the case of systems of evolution equations.

Definition 2: Two systems of equations

$$\begin{aligned} u_t^i &= u_{xxx}^i + f(x, t, u^i, u_x^i), \\ u_\sigma^i &= u_{\xi\xi\xi}^i + g(\xi, \sigma, u^i, u_\xi^i), \end{aligned} \quad (26)$$

will be called equivalent if there exists a change of variables of the form

$$\begin{aligned}\xi &= \alpha(t)x + \beta(t), & \sigma &= \gamma(t), \\ u^i(x, t) &= \Gamma(t)u^i(\xi(x, t), \sigma(x, t)) + \eta(x, t),\end{aligned}\tag{27}$$

which is invertible. The first result is given in the following statement.

Proposition 1: *The system*

$$\begin{aligned}u_t &= u_{xxx} + \frac{c_0}{\sqrt{t}}uu_x, \\ v_t &= v_{xxx} + \frac{c_1}{\sqrt{t}}(uv)_x,\end{aligned}\tag{28}$$

where c_0 and c_1 arbitrary constants, may be transformed into the autonomous perturbation of KdV system

$$\begin{aligned}u'_\sigma &= u'_{\xi\xi\xi} + c_0u'u'_\xi, \\ v'_\sigma &= v'_{\xi\xi\xi} + c_1(u'v')_\xi,\end{aligned}\tag{29}$$

through a transformation of the form (27) if and only if $c_0 = c_1$.

The validity of this Proposition allows us to state the following.

Proposition 2: *The non-autonomous JKdV system (21) is transformed into the autonomous JKdV system(10) through the transformation of the form*

$$\begin{aligned}u(x, t) &= t^{-1/2}u'(xt^{-1/2}, -2t^{-1/2}) - \frac{1}{2c_1}xt^{-1/2}, \\ v(x, t) &= t^{-1/2}v'(xt^{-1/2}, -2t^{-1/2}).\end{aligned}\tag{30}$$

Similar to Propositions 1 and 2 we have the following statement.

Proposition 3: *The non-autonomous JKdV system (23) is transformed into the the autonomous JKdV system (12) through the transformation*

$$\begin{aligned}
u(x, t) &= t^{-1/2} u'(xt^{-1/2}, -2t^{-1/2}) + \frac{1}{4c_0} xt^{-1/2}, \\
v(x, t) &= t^{-1/2} v'(xt^{-1/2}, -2t^{-1/2}), \\
w(x, t) &= t^{-1/2} w'(xt^{-1/2}, -2t^{-1/2}).
\end{aligned} \tag{31}$$

From the above discussions we have the following result.

Proposition 4: *The non-autonomous JKdV system (18) (or its extension (24)) can not be transformed into the JKdV system (9) (or its extension (13)) through a transformation of the form (27).*

We have observed that for some special cases of $N = 2$ and $N = 3$ time dependent systems transform to time independent cases. This comes indeed from the type of the Jordan algebra. For general N we have the following statement

Proposition 5. *A Jordan system (14) is equivalent to an autonomous Jordan system (1) if there exists an element \mathbf{a} of J such that $\mathbf{a}^2 = \mathbf{a}$ and $q \circ \mathbf{a} = q$ for all $q \in J$*

Proof. We write the system of equations (14) in the form $q_t = q_{xxx} + \frac{1}{\sqrt{t}} q \circ q_x$, where q takes values in a Jordan algebra J . Take the point transformation

$$\begin{aligned}
q(x, t) &= t^{-1/2} v(\xi, \tau) - \frac{1}{2} xt^{-1/2} \mathbf{a} \\
\xi &= xt^{-1/2}, \quad \tau = -2t^{-1/2}.
\end{aligned} \tag{32}$$

Then equations for v becomes time independent.

Transformable case in $N = 2$ (Example (2.ii)) is the case with $\mathbf{a} = e_1$ where $\{e_i, i = 1, 2\}$ are a basis of J . The Example (4.i) in $N = 3$ case is also transformable because the element $\mathbf{a} = -\frac{1}{2c_0} e_1$ satisfies the condition $\mathbf{a}^2 = \mathbf{a}$.

We would like to remark on the symmetries of (14). The first symmetry is the x -translational symmetry $\sigma_1^i = q_x^i$. The next one is the scale symmetry $\sigma_2^i = t q_t^i + \frac{1}{3} x q_x^i + \frac{1}{6} q^i$. The first generalized symmetry is given by $\sigma_3^i =$

$\mathcal{R}_j^i \sigma_2^j$, where \mathcal{R} is the recursion operator (17) of the system (14). This symmetry is nonlocal and contains the associator (tensor F_{jkl}^i) of the algebra J . There exists also an additional symmetry, the Galilean symmetry, $\eta_1^i = \sqrt{t} S_{jk}^i q_x^k k^j + \frac{1}{2} k^i$ for the system (14) satisfying $S_{jk}^i k^j = \delta_k^i$. Here we remark also that the element $\mathbf{k} = k^i e_i$ of J satisfies $\mathbf{k}^2 = \mathbf{k}$ hence due to the above proposition 5 the corresponding systems are transformable to autonomous KdV systems (1). In the general case, $F \neq 0$, $\sigma^i = \Lambda_j^i k^j$ is a symmetry of the non-autonomous JKdV system (14) for all k , where

$$\Lambda_j^i = \frac{1}{3} \sqrt{t} s_{jk}^i q_x^k + \frac{1}{6} \delta_j^i + \frac{1}{9} F_{lkj}^i q^l D^{-1} q^k. \quad (33)$$

In the case of time dependent recursion operators (and time dependent evolution equations) there is an ambiguity in calculating the symmetries. It is claimed that the recursion operators do not in general map symmetries to symmetries [17]. Following [17] time dependent higher symmetries can be constructed recursively by means of the extended recursion operator

$$\mathcal{R}_e = \mathcal{R} + \Lambda \int^t dt' \Pi D^2, \quad (34)$$

where \mathcal{R} is the recursion operator given in (17), and Π is the projection on the kernel of D defined explicitly by $\Pi f(t, x, q, q_x, \dots) = f(t, 0, 0, 0, \dots)$.

This work is partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK) and Turkish Academy of Sciences (TUBA).

References

- [1] P. J. Olver and V. V. Sokolov, *Commun. Math. Phys.*, **193**, (1998) 245.
- [2] M. Gürses, A. Karasu, and V. V. Sokolov, *J.Math.Phys.*, **40**, (1999) 6473.
- [3] P. J. Olver and J. P. Wang, *Proc. London. Math. Soc.*, **81**, (2000) 566.
- [4] S.I. Svinolupov, *Theor.Mat.Fiz.*, **87** (1991) 391.
- [5] P. J. Olver, *Applications of Lie Groups to Differential Equations* Second Edition, Graduate Texts in Mathematics, Vol.107, Springer-Verlag, New York, 1993.

- [6] S.I. Svinolupov , *Functional Anal.Appl.*, **27**(1994) 257.
- [7] C. Athorne and A. Fordy, *J. Phys. A. Math. Gen*, **20**, (1997) 1377.
- [8] F. Calogero and A. Degasperis ,*Lett. Nuovo Cim.*, **23**, (1978) 150
- [9] W. Oevel and A.S. Fokas, *J.Math.Phys.*, **25**(1984)918.
- [10] M. Gürses and A. Karasu, *Phys.lett. A* **214**(1996)21.
- [11] M. Gürses and A. Karasu, *J. Math. Phys.*, **39**, (1998) 2103.
- [12] G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer Verlag, New York -Berlin , 1989. See also the related references therein.
- [13] J.D. Kingston, *J.Phys.A: Math.Gen.*, **24**(1991) L769.
- [14] R. Hirota, *Phys.lett.A*, **71** (1979) 393.
- [15] L. Abellanas and A.Galindo ,*Phys.Lett.A*, **108**(1985)123.
- [16] B. Fuchssteiner, *J.Math.Phys.*, **34**(1993)5140.
- [17] J. A. Sanders and J. P. Wang, *Physica D*. **149**, (2001) 1.